PMM U.S.S.R.,Vol.45,pp.417-418
0021-8928/82/3 0417 $7.50/0

UDC 532.36

EQUILIBRIUM INSTABILITY IN A POTENTIAL FIELD, TAKING ACCOUNT OF VISCOUS FRICTION*

V.V. KOZLOV

The influence of dissipative forces on the equilibrium stability of natural mechanical systems is studied in this note. It is proved that if the analytic potential energy has no local minimum at equilibrium, then this equilibrium will be unstable after the addition of arbitrarily small dissipative forces.

1. Hypothesis on instability. Let us consider a natural mechanical system with n degrees of freedom whose generalized coordinates are $(x_1, \ldots, x_n) = \mathbf{x}$. Let $K(\mathbf{x}^\cdot, \mathbf{x}) = \Sigma a_{ij}(\mathbf{x}) x_i^\cdot x_j^\cdot$ be the kinetic, and $\Pi(x_1, \ldots, x_n)$ the potential energy of this system. The critical points of the function $\Pi(\mathbf{x})$ and only they are equilibrium positions. Everywhere beneath of $\mathbf{x} = \mathbf{0}$ is a critical point of function $\Pi(\mathbf{x})$ and $\Pi(0) = 0$. If the potential energy has a strict local minimum at the equilibrium position, then the equilibrium is stable (Lagrange theorem). There is an assumption that when the system is analytic (i.e., the functions $a_{ij}(\mathbf{x})$ and $\Pi(\mathbf{x})$ are analytic), and the potential energy has no local minimum at the equilibrium position, then the corresponding equilibrium state is unstable. An analogous assertion is apparently valid even for the infinitely differentiable case, however, as the known Painlevé—Wintner example shows

$$K = x^{\cdot 2}/2, \quad \Pi(x) = \exp x^{-2} \cos x^{-1} \quad (x \neq 0, \ \Pi(0) = 0)$$

in addition it is necessary to required isolation of the equilibrium position (or at least the absence of critical points of the function $\Pi(\mathbf{x})$ in the domain $\{x: \Pi(\mathbf{x}) < 0, \|\mathbf{x}\| < \varepsilon\}$ for small $\varepsilon > 0$). The proof of these assumptions is a complex problem, solved only in certain particular cases (see /1—3/, for example).

2. Instability of equilibrium subjected to viscous friction. Let us assume that the nonpotential forces $\mathbf{F}(\mathbf{x}, \mathbf{x}^\cdot)$ act on the system and $\mathbf{R}^n\{\mathbf{x}\} \times \mathbf{R}^n\{\mathbf{x}^\cdot\} \rightarrow \mathbf{R}^n$ are certain smooth vector functions. The equations of motion will have then the following form

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{x}^\cdot} - \frac{\partial L}{\partial \mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{x}^\cdot), \quad L = K - \Pi \tag{1}$$

We designate the nonpotential forces F the viscous friction forces if $\mathbf{F}(\mathbf{x}, 0) = 0$ and $E^\cdot < 0$ for $\mathbf{x}^\cdot \neq 0$; here $E = K + \Pi$ is the total system energy (compare with /4/, Ch. VIII). It is easy to verify that the equilibrium positions of the new mechanical systems will again agree with the critical points of the function $\Pi(\mathbf{x})$. The equilibrium states, which are stable according to the Lagrange theorem, here remain stable even upon the addition of viscous friction forces. In the next section we prove the following theorem.

Theorem. Let the point $\mathbf{x} = \mathbf{0}$ not be a local minimum of the function $\Pi(\mathbf{x})$. The equilibrium state $(\mathbf{x}, \mathbf{x}^\cdot) = (0, 0)$ of the system (1) is unstable if one of the following conditions is satisfied :

A) The function $\Pi(x_1, \ldots, x_n)$ is expanded in a convergent power series in $x_1, \ldots, x_n$ in the neighborhood of zero;

B) The function $\Pi(\mathbf{x})$ is infinitely differentiable in the neighborhood of zero and there are no critical points in the domain $\{\mathbf{x}: \Pi(\mathbf{x}) < 0, \|\mathbf{x}\| < \varepsilon\}$ for a certain $\varepsilon > 0$.

The formulation of the equilibrium stability problem in the presence of viscous friction and the assertion just formulated go back to Poincaré studies on the stability of the figures of equilibrium of a rotating fluid with energy dissipation taken into account /4/. This theorem can be considered as a generalization of known results of Kelvin /2,4/, Chetaev /2/, Salvadori /5/, and other authors about the influence of dissipative forces on the stability of equilibrium. We are not so much interested in the fact of the instability of the equilibrium position as on the mechanism of this phenomenon that will be explained during the proof of the theorem.

*Prikl.Matem.Mekhan.,45,No.3,570-572,1981

3. Proof of the theorem. Case B. Let us examine the motion $\mathbf{x}(t)$ with the following initial data: $\mathbf{x}(0)=\mathbf{x}_0$, $\mathbf{x}^{\cdot}(0)=0$; $\Pi(\mathbf{x}_0)<0$ and $\|\mathbf{x}_0\|\leqslant\varepsilon$. Let us prove the existence of a certain small number $\delta>0$ such that if $\|\mathbf{x}^{\cdot}(t)\|<\delta$ then $K^{\cdot\cdot}(\mathbf{x}(t),\mathbf{x}(t))\geqslant c_1>0$ (c_1, as well as the numbers $c_2,\ldots,c_5$ to be defined below, are independent of the time but dependent on ε and the initial conditions). To do this, we use the Legendre transformation $\mathbf{p}=\partial K/\partial\mathbf{x}^{\cdot}$ and the "canonical" equations

$$\mathbf{p}^{\cdot}=-\frac{\partial K}{\partial\mathbf{x}}-\frac{\partial\Pi}{\partial\mathbf{x}}+\mathbf{F}(\mathbf{x},\mathbf{p}),\quad \mathbf{x}^{\cdot}=\frac{\partial K}{\partial\mathbf{p}}$$

Evidently $\mathbf{F}(\mathbf{x},0)=0$. We first evaluate

$$K^{\cdot}=\frac{\partial K}{\partial\mathbf{p}}\mathbf{p}^{\cdot}+\frac{\partial K}{\partial\mathbf{x}}\mathbf{x}^{\cdot}=-\frac{\partial K}{\partial\mathbf{p}}\frac{\partial\Pi}{\partial\mathbf{x}}+\frac{\partial K}{\partial\mathbf{p}}\mathbf{F}(\mathbf{x},\mathbf{p})$$

Furthermore

$$K^{\cdot\cdot}=\left\langle\frac{\partial\Pi}{\partial\mathbf{x}},\ \frac{\partial^2K}{\partial\mathbf{p}^2}\frac{\partial\Pi}{\partial\mathbf{x}}\right\rangle+\Phi(\mathbf{x},\mathbf{p})$$

where $\langle,\rangle$ is the ordinary scalar product in $\mathbf{R}^n$, and the smooth function $\Phi(\mathbf{x},\mathbf{p})$ vanishes for $\mathbf{p}=0$. Since $E^{\cdot}(\mathbf{x},\mathbf{x}^{\cdot})\leqslant 0$ and $E(\mathbf{x}_0,\mathbf{x}_0^{\cdot})=\Pi(\mathbf{x}_0)<0$ then the trajectory of the motion $\mathbf{x}(t)$ lies in the domain $U=\{\mathbf{x}:\Pi(\mathbf{x})\leqslant\Pi(\mathbf{x}_0)<0\}$ for $t\geqslant 0$. There are no critical points of the function $\Pi(\mathbf{x})$ in this domain for $\|\mathbf{x}\|\leqslant\varepsilon$, the metric $K(\mathbf{x},\mathbf{p})$ is nondegenerate, and therefore, for small $\delta>0$ the estimate $K^{\cdot\cdot}\geqslant c_1>0$ will follow from the inequality $\|\mathbf{x}^{\cdot}\|<\delta$.

Since $E^{\cdot}=f(\mathbf{x},\mathbf{x}^{\cdot})=0$ only for $\mathbf{x}^{\cdot}=0$, but $f<0$ for the remaining values of the velocity, then for $\mathbf{x}\in U\cap\{\|\mathbf{x}\|<\varepsilon\}$ and $\delta/2\leqslant\|\mathbf{x}^{\cdot}\|\leqslant\varepsilon$ the function will be $f=E^{\cdot}\leqslant-c_2\,(c_2>0)$. We will prove that $E(\mathbf{x}(t),\mathbf{x}^{\cdot}(t))\to-\infty$ as $t\to+\infty$, if $\|\mathbf{x}(t)\|<\varepsilon$ and $\|\mathbf{x}^{\cdot}(t)\|<\varepsilon$. This contradiction will prove the instability of the equilibrium state $(\mathbf{x},\mathbf{x}^{\cdot})=(0,0)$.

Indeed for $\|\mathbf{x}(t)\|<\varepsilon$ and $\|\mathbf{x}^{\cdot}(t)\|<\varepsilon$ the function $|K^{\cdot}(t)|\leqslant c_3\ (c_3>0)$. If at a certain time $\|\mathbf{x}^{\cdot}\|<\delta/2$, then after a finite time interval (because of the estimate $|K^{\cdot}|\leqslant c_3, K^{\cdot\cdot}\geqslant c_1$) the quantity $\|\mathbf{x}^{\cdot}\|$ will not become less than δ. The time segment Δ when $\|\mathbf{x}^{\cdot}\|$ is increased from $\delta/2$ to δ, allows the estimate $\Delta\geqslant c_4>0$. During this time $E^{\cdot}\leqslant-c_2<0$ and, therefore, the function E is diminished by at least $c_5=c_2c_4>0$. Hence, if a sequence $\{t_k\}$, $t_k\to\infty\ (k\to\infty)$ such that $\|\mathbf{x}^{\cdot}(t_k)\|<\delta/2$, then evidently $E(t)\to-\infty$ as $t\to\infty$. If $\|\mathbf{x}^{\cdot}(t)\|\geqslant\delta/2$ starting with a certain time, then again $E(t)\to-\infty$ as $t\to\infty$.

The case A is derived from case B since for sufficiently small $\varepsilon>0$ there exists just the zero-th critical value of the analytic function $\Pi:\{\mathbf{x}:\|\mathbf{x}\|<\varepsilon\}\to\mathbf{R}$ (see /6/).

REFERENCES

1. LIAPUNOV A.M., General Problem on the Stability of Motion. ONTI, Main Publishers of General Engineering Literature, Moscow—Leningrad, 1935.
2. CHETAEV N.G., Stability of Motion. English translation, Pergamon Press, Book No. 09505, 1961.
3. PALAMODOV V.P., On the stability of equilibrium in a potential field. Funktsional'nyi Analiz i Ego Prilozheniia, Vol.11, No.4, 1977.
4. APPEL P., Figures of Equilibrium of a Rotating Homogeneous Fluid. ONTI, Main Publishers of General Engineering Literature, Moscow—Leningrad, 1936.
5. SALVADORI L., Sull' estensione ai sistemi dissipativi del criterio di stabilità del Routh, Ricerche Mat., Vol.15, No.2, 1968.
6. SOUČEK J. and SOUČEK V., Morse—Sard theorem for real analytic functions. Comment. Math. Univ. Carolinae, Vol.13, No.1, 1973.

Translated by M.D.F.